TITLE: Macroscopic Models of Radiative Transfer as Applied to Computation of the Radiation Field in the Solar Atmosphere

Macroscopic models of radiative transfer as applied to computation of the radiation field in the solar atmosphere

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1. Motivation and objectives

The *two stream method*, also called the *two-flux approximation*, distinguishes between incoming and outgoing radiation for improved accuracy relative to one-stream methods while retaining reduced numerical cost relative to full RTE solutions (Mihalas & Mihalas 1984; Siegel & Howell 2001). It assumes different radiation from each source, each uniform over its half-space, which constitutes the weak point of the method.

This concept has recently been used by Dubroca & Klar (2002), where they derive a unidimensional macroscopic moment radiation model for each stream; they call this a half moment model. The pressure closure is obtained by using maximization of entropy (Minerbo 1978), which allows them to avoid any isotropic assumption about the radiative intensities.

In this paper a new three dimensional half-moment model for radiative transfer is presented for a gray medium. It describes the evolution of the zeroth and first directional half moments of the radiative intensity. The closure is provided, similarly to Dubroca & Klar (2002), by the maximum entropy concept. This work generalizes that model to three dimensions.

The splitting of the direction of propagation $\mathbf{\Omega}$ into two pieces, $\mathbf{\Omega}^+$ and $\mathbf{\Omega}^-$, in Dubroca & Klar (2002) was done by cutting the $\mathbf{\Omega}$-space in a static sense, meaning that the same definition of $+$ and $-$ was used at all points in the domain. This direction splitting is clearly the best, at least the most intuitive one, for unidimensional problems, but this is not necessarily true for multi-dimensional problems. As a matter of fact, and in contrast, the splitting is here done dynamically according to the direction of the total radiative flux at each point. At any point of the domain, our model considers that the radiative flux defines the main direction of propagation, the positive direction $\mathbf{\Omega}^+$, and a negative one, in the opposite direction, $\mathbf{\Omega}^-$. This dynamic way of splitting the domain of directions appears to be a natural one for multidimensional problems.

This particular choice for the splitting also allows the pressure model to be analytically computed, which is not the case for a static definition of the half moments. However, it does have a very unfortunate consequence: since the half spaces $\mathbf{\Omega}^+$ and $\mathbf{\Omega}^-$ are dependent on the radiative flux, they become variable in time and space. The integration of the radiative transfer equation (RTE) over these subspaces is then complicated. Nevertheless, if the radiation is assumed isotropic in the plane perpendicular to the direction of propagation, which we believe is a reasonable assumption for a two-direction model, the integration of the RTE over these spaces can be done. Unfortunately, the integration introduces unclosed border terms involving the intensity in this plane. A closure will then be provided below by a model for this intensity.

In order to derive the radiative pressure tensor and then to be able to close the system,

a further assumption is necessary: the positive component of the flux is assumed to be nearly parallel to the total flux, the negative one becoming then necessarily anti-parallel. This is obviously true in one dimension, but not in general, and constitutes our main assumption in the pressure derivation. It holds nevertheless exactly in two important limits: at radiative equilibrium and for strongly anisotropic radiation.

The model presented here (the derivation being done in Ripoll & Wray (2003)), called the $M_1^{1/2}$ model, is a hyperbolic system consisting of a total of eight equations in three dimensions, four equations for each direction. Each half model has the classical form of a macroscopic moment model in which the pressure tensor is constructed from the well-known Eddington tensor with a particular Eddington factor. Moreover, different source and border terms occur. The latter introduce couplings between the macroscopic and microscopic quantities and between the $+$ and $-$ streams, through the intensity in the plane perpendicular to the flux.

One of the major advantages of this model is that independent incoming and outgoing boundary conditions are allowed, which is not possible with full moment models. Moreover, the flux stays limited by the speed of light and the underlying intensity, which can always be deduced from the macroscopic quantities. The flux is described by a Planck function at radiative equilibrium or by one (or two) Dirac function(s) in one (or two) direction(s) of propagation in the anisotropic limit(s). Furthermore, the maximum entropy closure, which has been often applied to radiative transfer (see for instance (Ripoll 2004)) will be shown as a very useful and powerful concept allowing the derivation of new accurate, well-defined, and robust models.

The main theoretical application of the half moment model, treated in this paper, is its reduction to a full moment model, called M_1^+, for the particular but important case of a hot, opaque source radiating in a cold transparent (or semi-transparent) medium for very specific applications, such as stellar interiors or atmospheres, or combustion problems. This model consists of four equations and is derived from the half moment model with fairly simple arguments. The model is tested on a simple test case for different values of the opacity and will be shown to give very good results, better than those obtained from either the P_1 or M_1 closures. For all problems presented in this paper, the solutions obtained by the new models are compared with those obtained by using a ray-tracing solver of the RTE. It will be shown numerically, partially here and mostly in Ripoll & Wray (2003), that the M_1^+ model constitutes an improvement of the existing closures and may be particularly useful for treating radiation in stellar interiors or atmospheres.

The problem of a modeled solar atmosphere, in which the opacity is roughly approximated by that at a wavelength of $500\,nm$ (Vernazza *et al.* 1981), will be solved and discussed. The radiation field is obtained from the M_1^+ model at reasonable accuracy.

The structure of the paper is as follows. In section 2, the model $M_1^{1/2}$ is presented. In section 3, for the particular case of a hot, opaque source radiating into a cold medium, the half moment model is reduced to the M_1^+ model. In section 4, we first solve a simple and academic problem to validate the models, followed by a simplified solar atmosphere.

2. The Half-Moment Model

2.1. *The radiative transfer equation*

The radiative transfer equation in an non-scattering, emitting, and absorbing gray medium is given by

$$\frac{1}{c}\partial_t I + \mathbf{\Omega} \cdot \nabla I = \sigma \mathcal{B}(\nu, T) - \sigma I, \tag{2.1}$$

where the intensity $I = I(t, \mathbf{r}, \mathbf{\Omega}, \nu)$ is a function of the time t, the position $\mathbf{r}$, the direction of propagation $\mathbf{\Omega}$ and the frequency ν. The Planck radiative intensity $\mathcal{B}$ describes the isotropic emission of the medium at frequency ν and temperature T. Here c is the velocity of light and σ, the spectral absorption coefficient or opacity, is assumed to be independent of ν. By integrating the RTE over frequency and introducing the quantity $J(t, \mathbf{r}, \mathbf{\Omega}) = \int_0^\infty I(t, \mathbf{r}, \mathbf{\Omega}, \nu)\, d\nu \equiv \langle\, I \,\rangle_\nu$ †, we obtain

$$\frac{1}{c}\partial_t J + \mathbf{\Omega} \cdot \nabla J = \frac{\sigma a}{4\pi} T^4 \; - \sigma J, \tag{2.2}$$

where the constant a is given by $a = (8\pi^5 k^4)/(15 h^3 c^3)$.

2.2. *Derivation of the Half-Moment Model*

Two half spaces, defined by the direction of propagation of the radiation, are introduced, splitting the domain $\mathbf{\Omega}$ into 2 disjoint pieces, $\mathbf{\Omega}^+$ and $\mathbf{\Omega}^-$, such that $\mathbf{\Omega} = \mathbf{\Omega}^+ \cup \mathbf{\Omega}^-$, $\mathbf{\Omega}^+ \cap \mathbf{\Omega}^- = \emptyset$, and

$$\mathbf{\Omega}^+ = \{\mathbf{\Omega} \;/\; \mathbf{F}_R/F_R \cdot \mathbf{\Omega} \geq 0\}, \tag{2.3}$$

$$\mathbf{\Omega}^- = \{\mathbf{\Omega} \;/\; \mathbf{F}_R/F_R \cdot \mathbf{\Omega} < 0\}, \tag{2.4}$$

where $\mathbf{F}_R$ designates the radiative flux and F_R its norm‡. $\mathbf{\Omega}^\pm$ are then two unit half spheres, obtained by cutting the sphere $\mathbf{\Omega}$ by the plane perpendicular to the flux vector $\mathbf{F}_R$, $P_\perp = \{\Omega \;/\; \mathbf{F}_R \cdot \mathbf{\Omega} = 0\}$.

The three first moments, E_R, $\mathbf{F}_R$, and $\mathbf{P}_R$, respectively the radiative energy, flux vector and pressure tensor, of the radiative intensity I according to the frequency and the direction are defined by

$$E_R = \frac{1}{c}\langle\, J \,\rangle_\Omega \quad \mathbf{F}_R = \langle\, \mathbf{\Omega}\, J \,\rangle_\Omega, \quad \mathbf{P}_R = \frac{1}{c}\langle\, \mathbf{\Omega} \otimes \mathbf{\Omega}\, J \,\rangle_\Omega, \tag{2.5}$$

In a similar way, the half moments $E_R{}^\pm$, $\mathbf{F}_R^\pm$, and $\mathbf{P}_R^\pm$ are defined by

$$E_R{}^\pm = \frac{1}{c}\langle\, J \,\rangle_{\mathbf{\Omega}^\pm}, \quad \mathbf{F}_R^\pm = \langle\, \mathbf{\Omega}\, J \,\rangle_{\mathbf{\Omega}^\pm}, \quad \mathbf{P}_R^\pm = \frac{1}{c}\langle\, \mathbf{\Omega} \otimes \mathbf{\Omega}\, J \,\rangle_{\mathbf{\Omega}^\pm}, \tag{2.6}$$

By construction, the following properties hold

$$E_R = E_R{}^+ + E_R{}^-, \;\; \mathbf{F}_R = \mathbf{F}_R^+ + \mathbf{F}_R^-, \;\; \mathbf{P}_R = \mathbf{P}_R^+ + \mathbf{P}_R^-. \tag{2.7}$$

Three unit vectors of propagation $\mathbf{n}$, $\mathbf{n}^+$, $\mathbf{n}^-$ are defined according to the direction of the radiative flux and half fluxes:

$$\mathbf{n} = \mathbf{F}_R/F_R, \quad \mathbf{n}^+ = \mathbf{F}_R^+/F_R^+, \quad \mathbf{n}^- = \mathbf{F}_R^-/F_R^-, \tag{2.8}$$

It is assumed herein that the positive component of the flux can be taken to be approximately parallel to the flux itself, $\mathbf{n}^\pm \simeq \pm\mathbf{n}$ (hypothesis **H1**). This assumption is true

† We will denote the integration of a function f over the variables X, Y, Z as $\langle\, f \rangle_{X,Y,Z}$
‡ for the sake of brevity, the norm of a vector $\mathbf{v}$ will be always denoted as v.

when the radiation is isotropic or strongly anisotropic but is less accurate in intermediate cases. As a consequence of the previous definitions, the negative component of the flux then becomes anti-parallel to the flux $\mathbf{F}_R$. The previous relationship among the radiative fluxes can then be rewritten as

$$\mathbf{F}_R = F_R^+ \mathbf{n}^+ + F_R^- \mathbf{n}^- \simeq F_R^+ \mathbf{n} - F_R^- \mathbf{n}. \tag{2.9}$$

Integrating the radiative transfer equation (2.2) over each half direction $\mathbf{\Omega}^\pm$, leads to

$$\int_{\mathbf{\Omega}^\pm} \partial_t J \, d\mathbf{\Omega} + \int_{\mathbf{\Omega}^\pm} \mathbf{\Omega} \cdot \nabla J \, d\mathbf{\Omega} = \frac{1}{2} c\sigma a T^4 - c\sigma E_R^\pm. \tag{2.10}$$

Multiplying the radiative transfer equation (2.2) by $\mathbf{\Omega}$ and integrating over the half directions $\mathbf{\Omega}^\pm$ leads to

$$\frac{1}{c} \int_{\mathbf{\Omega}^\pm} \partial_t \mathbf{\Omega} J \, d\mathbf{\Omega} + c \int_{\mathbf{\Omega}^\pm} \mathbf{\Omega} \otimes \mathbf{\Omega} \cdot \nabla J \, d\mathbf{\Omega} = \frac{1}{4} c\sigma a T^4 \mathbf{n}^\pm - \sigma \mathbf{F}_R^\pm. \tag{2.11}$$

The two subspaces $\mathbf{\Omega}^\pm$ have been chosen in order to partition the domain in a physically natural way and to allow the radiative pressure to be closed (see section 2.3). But the partitioning causes the bounds of integration over $\mathbf{\Omega}^\pm$ to depend on the flux and hence on x, y, z, t, and so disallows commuting derivatives and integrals over $\mathbf{\Omega}^\pm$. Nevertheless, it has been possible to perform these integrals. The following relationships hold provided that the radiation is assumed to be isotropic in the plane perpendicular to the direction of the flux (hypothesis **H2**), for example if described or modeled by a radiative intensity $I_\perp$ such as that in the next section.

$$\int_{\mathbf{\Omega}^\pm} \partial_t J \, d\mathbf{\Omega} \simeq \partial_t E_R^\pm, \tag{2.12}$$

$$\int_{\mathbf{\Omega}^\pm} \mathbf{\Omega} \cdot \nabla J \, d\mathbf{\Omega} \simeq \nabla \cdot \mathbf{F}_R^\pm - \pi J_\perp \nabla \cdot \mathbf{n}^\pm, \tag{2.13}$$

$$\int_{\mathbf{\Omega}^\pm} \partial_t \mathbf{\Omega} J \, d\mathbf{\Omega} \simeq \partial_t \mathbf{F}_R^\pm - \frac{\pi}{c} J_\perp \partial_t \mathbf{n}^\pm, \tag{2.14}$$

$$\int_{\mathbf{\Omega}^\pm} \mathbf{\Omega} \otimes \mathbf{\Omega} \cdot \nabla J \, d\mathbf{\Omega} \simeq \nabla \cdot \mathbf{P}_R^\pm, \tag{2.15}$$

where $J_\perp = \langle I_\perp \rangle_\nu$. These relationships are derived in Ripoll & Wray (2003) using assumption (**H2**) which leads to canceling of two of the border terms in (2.12) and in (2.15) and to a simple form of (2.13) and (2.14). Thus, the $M_1^{1/2}$ model is given in three dimension by

$$\partial_t E_R^\pm + \nabla \cdot \mathbf{F}_R^\pm - \pi J_\perp \nabla \cdot \mathbf{n}^\pm = \frac{1}{2} c\sigma a T^4 - c\sigma E_R^\pm, \tag{2.16}$$

$$\frac{1}{c} \partial_t \mathbf{F}_R^\pm - \frac{\pi}{c} J_\perp \partial_t \mathbf{n}^\pm + c \nabla \cdot \mathbf{P}_R^\pm = \frac{1}{4} c\sigma a T^4 \mathbf{n}^\pm - \sigma \mathbf{F}_R^\pm. \tag{2.17}$$

The closure of the model will be provided by models for the radiative pressure and $J_\perp$ in the next section.

2.3. *Closure of the Half-Moment Model*

2.3.1. *A model for the pressure*

A special radiative intensity I^* with the following form is chosen

$$I^*(t,\mathbf{r},\boldsymbol{\Omega},\nu) = I(T^*(t,\mathbf{r},\boldsymbol{\Omega}),\nu) = \frac{2h\nu^3}{c^2}\left[\exp\left(\frac{h\nu}{kT^*}\right)-1\right]^{-1}, \tag{2.18}$$

in which

$$T^*(t,\mathbf{r},\boldsymbol{\Omega}) = \frac{T}{B(1-\mathbf{A}\cdot\boldsymbol{\Omega})}, \tag{2.19}$$

and where $\mathbf{A}$ and B are defined on $\boldsymbol{\Omega}^+$ and $\boldsymbol{\Omega}^-$ as follows

$$\mathbf{A} = \left| \begin{array}{lll} A^+\mathbf{n}^+ & \text{on} & \mathbf{n}\cdot\boldsymbol{\Omega}\geq 0 \\ A^-\mathbf{n}^- & \text{on} & \mathbf{n}\cdot\boldsymbol{\Omega}< 0 \end{array}\right. \quad \text{and} \quad B = \left| \begin{array}{lll} B^+ & \text{on} & \mathbf{n}\cdot\boldsymbol{\Omega}\geq 0 \\ B^- & \text{on} & \mathbf{n}\cdot\boldsymbol{\Omega}< 0 \end{array}\right. . \tag{2.20}$$

This intensity, determined by the so-called maximum entropy closure, provides a way to close the radiative pressure $\mathbf{P}_R$ by approximating it with $\mathbf{P}_R^*$, computed from I^*. This constitutes the third and last assumption (**H3**) used here to derive the $M_1^{1/2}$ model.

If the scalar B and the vector $\mathbf{A}$ are defined from the two constraints $E_R = \langle I^*\rangle_{\nu,\boldsymbol{\Omega}}$ and $\mathbf{F}_R = \langle \boldsymbol{\Omega}\, I^*\rangle_{\nu,\boldsymbol{\Omega}}$, then the intensity I^*, defined in (2.18), maximizes the radiative entropy under these constraints (Minerbo 1978). The subdivision into $\boldsymbol{\Omega}^+$ and $\boldsymbol{\Omega}^-$ does not change this property, and as a direct consequence of (2.7) we have that the restricted intensities $I^*_{|\boldsymbol{\Omega}^\pm}$ maximize the radiative entropy under the constraints ${E_R}^\pm = \langle I^*\rangle_{\nu,\boldsymbol{\Omega}^\pm}$ and $\mathbf{F}_R^\pm = \langle \boldsymbol{\Omega} I^*\rangle_{\nu,\boldsymbol{\Omega}^\pm}$. Unfortunately, it is not possible to obtain a closed form for the pressure using these constraints directly; instead we use **H1** to approximate the constraints as follows

$$\left\langle I^*(\mathbf{n}^\pm)\right\rangle_{\nu,\boldsymbol{\Omega}^\pm} \simeq \langle I^*(\pm\mathbf{n})\rangle_{\nu,\boldsymbol{\Omega}^\pm} \quad \text{and} \quad \left\langle I^*(\mathbf{n}^\pm)\right\rangle_{\nu,\boldsymbol{\Omega}^\pm} \simeq \langle \boldsymbol{\Omega} I^*(\pm\mathbf{n})\rangle_{\nu,\boldsymbol{\Omega}^\pm}, \tag{2.21}$$

The abbreviated notation $I^*(\mathbf{n}^\pm)$ is used to denote I^* as defined using $\mathbf{A}$ in (2.20), and $I^*(\pm\mathbf{n})$ denotes I^* using instead

$$\mathbf{A} = A^+\mathbf{n}^+ \simeq A^+\mathbf{n} \quad \text{on} \quad \mathbf{n}\cdot\boldsymbol{\Omega}\geq 0 \quad ; \quad \mathbf{A} = A^-\mathbf{n}^- \simeq -A^-\mathbf{n} \quad \text{on} \quad \mathbf{n}\cdot\boldsymbol{\Omega}< 0, \tag{2.22}$$

which corresponds to the use of **H1** in the definition (2.20). The computation of $\mathbf{A}$, from now approximated by (2.22), and B is shown in done in Ripoll & Wray (2003). Actually, the assumption **H1** was not needed before this point of the derivation.

The radiative pressure tensor $\mathbf{P}_R^\pm$ is approximated by ${\mathbf{P}_R^\pm}^*$, computed in Ripoll & Wray (2003), and is written

$$\begin{aligned} \mathbf{P}_R^\pm &= \int_{\boldsymbol{\Omega}^\pm} \boldsymbol{\Omega}\otimes\boldsymbol{\Omega} J\, d\boldsymbol{\Omega} \simeq \int_{\boldsymbol{\Omega}^\pm} \boldsymbol{\Omega}\otimes\boldsymbol{\Omega} J^*(\mathbf{n}^\pm)\, d\boldsymbol{\Omega} = {\mathbf{P}_R^\pm}^* \simeq \int_{\boldsymbol{\Omega}^\pm} \boldsymbol{\Omega}\otimes\boldsymbol{\Omega} J^*(\pm\mathbf{n})\, d\boldsymbol{\Omega} \\ &\simeq \mathbf{P}_R^*(\mathbf{f}^\pm, {E_R}^\pm) = \mathbf{D}_R^\pm {E_R}^\pm = \mathbf{D}_R(\mathbf{f}^\pm){E_R}^\pm \end{aligned}$$

where $\mathbf{D}_R^\pm$ has the form of the well-known Eddington tensor evaluated for an anisotropy $\mathbf{f}^\pm = \mathbf{F}_R^\pm/(c{E_R}^\pm)$, given by

$$\mathbf{D}_R^\pm = \mathbf{D}_R(\mathbf{f}^\pm) = \frac{1-\chi(f^\pm)}{2}\mathbf{Id} + \frac{3\chi(f^\pm)-1}{2}\frac{\mathbf{f}^\pm\otimes\mathbf{f}^\pm}{{f^\pm}^2}, \tag{2.23}$$

in which the Eddington factor χ is

$$\chi(f^{\pm}) = \frac{8f^{\pm 2}}{1+6\,f^{\pm}+\sqrt{1+12\,f^{\pm}-12\,f^{\pm 2}}} \text{ if } f^{\pm} > 0.5 \text{ and } \chi(f^{\pm}) = 1/3 \text{ elsewhere.} \tag{2.24}$$

There is then no coupling between outgoing and incoming radiation in the pressure model. The isotropic and anisotropic limits ($f \to 0$ and $f \to 1$) of the model, which define its range of validity are discussed in Ripoll & Wray (2003). The corresponding range of $f^{\pm}$ is $[1/2, 1]$; this range will hold numerically as well if the limitation on the Eddington factor χ is enforced as in (2.24), but not otherwise. The Eddington factor is hence in total agreement with the domain of definition of $f^{\pm}$. Without this limitation on χ, $f^{\pm}$ lower than $1/2$ implies a radiative intensity defined, problematically, in terms of an $\mathbf{A}$ with a negative norm.

2.3.2. *A model for $I_\perp$ and $J_\perp$*

The radiative intensity I^*, which has been used to derive mean absorption coefficient (Ripoll *et al.* 2001), can be used as well to model the border terms coming from the integration, since I^* is isotropic in the plane perpendicular to the flux and we have

$$I_\perp^{*\,\pm} = \frac{2h\nu^3}{c^2}\left[\exp(\frac{h\nu T}{kB^{\pm}}) - 1\right]^{-1}, \tag{2.25}$$

since $\mathbf{A}^{\pm}\cdot\mathbf{\Omega} = 0$ for $\mathbf{\Omega} \in P_\perp$ and $B^{\pm}$ is given in Ripoll & Wray (2003). $I_\perp^{*\,\pm}$ is a Planck function evaluated in $T_R^{\pm}$ when radiation are isotropic and vanishes when radiation are anisotropic in the direction of the flux. The main problem of this model is that the intensity is reconstructed from the plus and minus macroscopic quantities. There is then two intensities $I_\perp^{*\,\pm}$, which should have the same value in $P_\perp$, but nothing can guaranteed it. The definition of $I_\perp^*$ in $P_\perp$ in then not obvious and we propose simply an average of the two values $I_\perp^{*\,\pm}$.

$$I_\perp \simeq I_\perp^* \simeq \frac{I_\perp^{*\,+} + I_\perp^{*\,-}}{2} \quad ; \quad J_\perp \simeq J_\perp^* \simeq \frac{J_\perp^{*\,+} + J_\perp^{*\,-}}{2}, \tag{2.26}$$

where $J_\perp^{*\,\pm}$ is given by

$$J_\perp^{*\,\pm} = \frac{aT^4}{4\pi B^{\pm 4}} = \frac{aT_R^{\pm 4}}{4\pi}\frac{6(1-A^{\pm})^3}{3-3A^{\pm}+A^{\pm 2}} \tag{2.27}$$

$$= \frac{3}{16\pi}aT_R^{\pm 4}\frac{\left(1-2f^{\pm}+\sqrt{-12f^{\pm 2}+12f^{\pm}+1)}\right)^3}{f^{\pm}(6f^{\pm}+1+\sqrt{-12f^{\pm 2}+12f^{\pm}+1)}}, \tag{2.28}$$

$J_\perp^{*\,\pm}$ is a function of $T_R^{\pm}$ and $f^{\pm}$ which decrease from their isotropic values $J_\perp^{*\,\pm^{iso}} = aT_R^{\pm 4}/2\pi$ (note that $J_\perp^{*\,\pm^{eq}} = aT^4/4\pi$ at equilibrium) to their anisotropic values $J_\perp^{*\,\pm^{aniso}} = 0$. Moreover, it can be seen that $J_\perp^{*\,iso} = E_R/4\pi$ could constitute a rough model for $J_\perp$.

One could think the average (2.26) is arbitrary and any average of the form $J_\perp^* \simeq (aJ_\perp^{*\,+} + bJ_\perp^{*\,-})/(a+b)$ could have been chosen. Actually, $a = 1$ and $b = 1$ is the only combination allowing the model (2.26) to get the exact isotropic value $J_\perp = E_R/(4\pi) = (E_R^+ + E_R^-)/(4\pi)$. (The demonstration is obvious using (2.28) with $f^{\pm} \to 1/2$ in (2.26)).

Finally, the 8 equations in three dimension of the $M_1^{1/2}$ model are

$$\partial_t E_R^{\pm} + \nabla \cdot \mathbf{F}_R^{\pm} - \pi J_\perp^* \nabla \cdot \mathbf{n}^{\pm} = \frac{1}{2} c\sigma a T^4 - c\sigma E_R^{\pm}, \tag{2.29}$$

$$\frac{1}{c}\partial_t \mathbf{F}_R^{\pm} - \frac{\pi}{c} J_\perp^* \partial_t \mathbf{n}^{\pm} + c\nabla \cdot \left(\mathbf{D}_R(\mathbf{f}^{\pm})\, E_R{}^{\pm}\right) = \frac{1}{4} c\sigma a T^4 \mathbf{n}^{\pm} - \sigma \mathbf{F}_R^{\pm}, \tag{2.30}$$

and have been derived using the three assumptions **H1**, **H2**, and **H3**. The positive and negative quantities are then only coupled by the model for $J_\perp^*$ given in (2.26)-(2.28), which constitutes the last approximation, and when the full moment are reconstructed.

3. A new 3D moment model for a hot opaque medium emitting in a cold medium

From the half moment model, a new moment model is derived for a particular, but nevertheless important, case: a hot opaque source emitting in a cold medium. It can be seen in (2.29)-(2.30) that the non-linearity of the radiative pressure does not allow reducing a closed full moment model from the sum of the two half-moment models in the general case. As a matter of fact, this sum leads to

$$\partial_t E_R + \nabla \cdot \mathbf{F}_R = c\sigma a T^4 - c\sigma E_R, \tag{3.1}$$

$$\frac{1}{c}\partial_t \mathbf{F}_R + c\nabla \cdot (\mathbf{D}_R(\mathbf{f}^+) E_R{}^+ + \mathbf{D}_R(\mathbf{f}^-) E_R{}^-) = -\sigma \mathbf{F}_R, \tag{3.2}$$

where the border terms have canceled when added, as did the half emission terms of the flux equations, and where a pressure remains which is expressed in terms of unclosed quantities.

From this system, it is however possible to reconstruct a full closed moment model in the particular case of a hot, opaque opaque source emitting in a cold, transparent (or semi-transparent) medium. We will discuss at the end of this section how the source and the exterior medium must be characterized with regard to their temperature and opacity.

We now discuss the two domains in this particular problem.

1) First, inside the hot, opaque source where radiation can be considered as nearly isotropic ($f^{\pm} \simeq 0.5$), the linear P_1 limit holds for both half-moment models since in this limit

$$\mathbf{P}_R \simeq \lim_{f^{\pm} \to 1/2} (\mathbf{D}_R^+ E_R^+ + \mathbf{D}_R^- E_R^-) = \frac{1}{3}(E_R^+ + E_R^-) = \frac{1}{3} E_R \tag{3.3}$$

The condition $f^{\pm} \simeq 0.5$ implies $f \simeq 0$. We will assume for this model that radiation can be considered isotropic for $f < 0.5$. It should be noticed that radiation is usually considered nearly isotropic for $f < 0.3$ (the P_1 validity domain); we assume that it is possible to extend this range to $f < 0.5$ without too much impact on the solution because, for many applications, the anisotropic factor f is predominantly either close to 0 or 1. The latter case will be handled next.

2) Within the cold, transparent medium, a cold equilibrium is assumed for the negative half moments while the positive ones are strongly anisotropic due to the radiating source, implying $f^+ >> 0.5$.

The exterior medium is assumed to be cold enough such that $E_R^- << E_R^+$ and $\|F_{R_i}{}^-\| << \|F_{R_i}{}^+\|$ (or indifferently $F_{R_i}{}^- << F_{R_i}{}^+$ $\forall i = 1..3$). Then $E_R \simeq E_R^+$ and $F_R \simeq F_R^+$ lead to $f \simeq f^+$ and hence $f >> 0.5$. Finally, using these approximations

$$\mathbf{P}_R = \mathbf{D}_R{}^+ E_R^+ + \mathbf{D}_R{}^- E_R^- \simeq \mathbf{D}_R(f^+) E_R^+ \simeq \mathbf{D}_R^+(f) E_R. \tag{3.4}$$

It must be noticed now that the Eddington factor $\mathbf{D}_R^+(f)$, given in (2.23)-(2.24) is limited by construction to $D_R^+ = 1/3$ when $f^+ < 0.5$, and since $f^+ \simeq f$, this leads to the P_1 closure when radiation is isotropic. This tensor then naturally makes the transition from $D_R^+ = 1/3$ to $D_R^+(f)$. Thus the same full moment model is found to be valid inside the hot, opaque medium and the cold one and is able to establish the transition. The radiation field is hence fully described by this model, called here M_1^+, which is written

$$\partial_t E_R + \nabla \cdot \mathbf{F}_R = c\sigma a T^4 - c\sigma E_R, \tag{3.5}$$

$$\frac{1}{c}\partial_t \mathbf{F}_R + c\nabla \cdot \left(\mathbf{D}_R^+(f)\, E_R\right) = -\sigma \mathbf{F}_R, \tag{3.6}$$

with

$$\mathbf{D}_R^+(f) = \frac{1-\chi^+(f)}{2}\,\mathbf{Id} + \frac{3\chi^+(f)-1}{2}\,\frac{\mathbf{f}\otimes\mathbf{f}}{f^2} \tag{3.7}$$

and

$$\chi^+(f) = \frac{8f^2}{1+6f+\sqrt{1+12f-12f^2}} \text{ if } f > 0.5 \text{ and } \chi(f) = 1/3 \text{ elsewhere.} \tag{3.8}$$

This four equation model gives equations for the full moments and for a range of the anisotropy $f \in [0,1]$. It should be noticed here that this reduction to a full-moment system is only possible because the positive flux always radiates in the full flux direction, whatever this direction is. With a classical splitting of the flux, where the positive component of the flux is statically defined, such a reduction is not possible since incoming and outgoing radiation cannot be distinguished.

This model describes radiation emitted by a source which is isotropic inside (P_1 must be valid inside) and radiating into a exterior domain in a strongly anisotropic way. This signifies that this exterior medium is not radiating much compared to the source, but might for example absorb part of emitting radiation of the source provided that it does not re-radiate towards the source. Moreover, if the exterior medium is too opaque or too hot, the anisotropy factor will stay lower than 0.5 and the model will simply reduce to P_1.

The domain of validity of M_1^+ is then quite large. It will be valid in many applications where the main interest is to compute the heat loss of a source by radiation in a non-remitting medium, in particular for luminous flames and fires burning in ambient air and for stars radiating into their thin atmospheres.

We will see in the next section that this model gives results very close to those obtained by a RTE ray-tracing solver.

4. Numerical Results

4.1. *A black hot sphere radiating into an infinite semi-transparent cold gas*

The first case is a hot black sphere of radius $r = 1$ which emits at an intensity $I_s = 1$ into an infinite absorbing cold gas ($T = 0$) of two different and small opacities $\sigma = 0.1, 0.01$. In terms of macroscopic quantities, we have inside the sphere: $E_{R_s}^{\pm} = 2\pi$, $\mathbf{F}_{R_s}^{\pm} = \pm\pi$, $E_{R_s} = 4\pi$ and $\mathbf{F}_{R_s} = 0$.

For this case the positive flux will remain positive and the negative quantities, which vanish, do not need to be computed. Moreover, the exact value of $J_\perp$, which here is simply $J_\perp = 0$, is used. The outgoing boundary conditions are $\partial_r {E_R}^+ = 0$, $\partial_r \mathbf{F}_R^+ = 0$ at

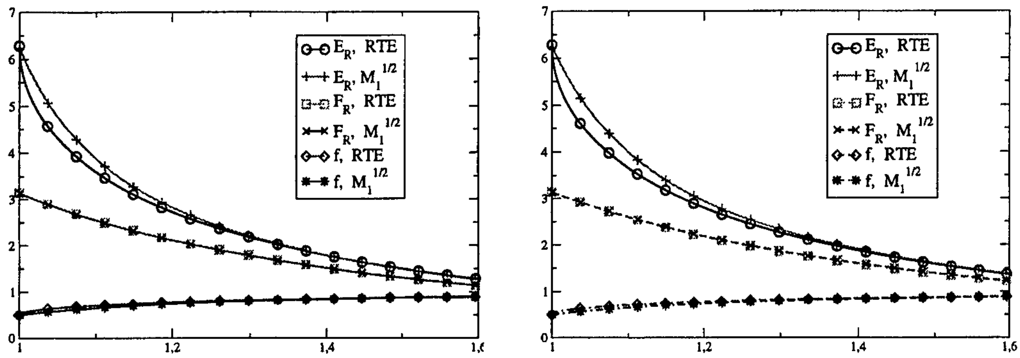

FIGURE 1. Radiative quantities: energy, flux and anisotropy. Left: $\sigma = 0.1$. Right: $\sigma = 0.01$

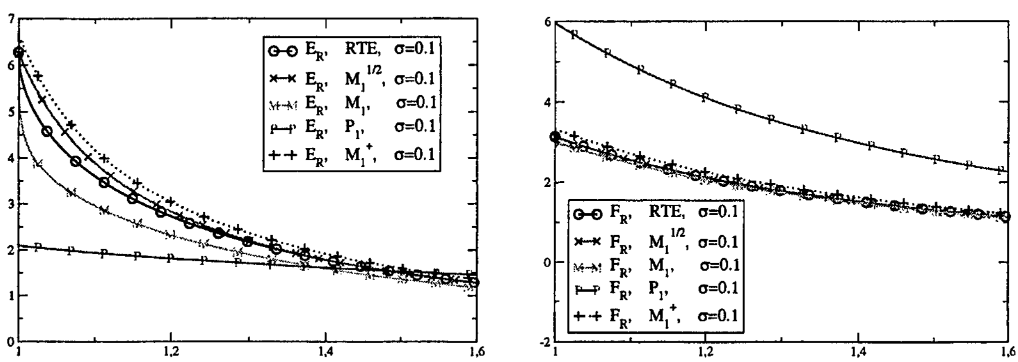

FIGURE 2. Comparison between 5 models for $\sigma = 0.1$. Left: Radiative Energy. Right: Radiative Flux.

$r = 1.6$, similarly $\partial_r E_R = 0$, $\partial_r \mathbf{F}_R = 0$ for the full moment models, in order to simulate an infinite domain.

We compare in Fig. 1, the steady states of the radiative energy, flux, and anisotropic factor given by $M_1^{1/2}$ to those obtained by the RTE solver for $\sigma = 0.1, 0.01$. Results are in very good agreement for each of these quantities.

We compare in Fig. 3-2, both radiative energy and flux obtained by five models for these opacities: RTE, $M_1^{1/2}$, M_1, P_1 and the new M_1^+. Results obtained by the RTE and the $M_1^{1/2}$ solvers were presented in the two previous figures. The full moment methods M_1, P_1 and M_1^+ are solved from inside the hot sphere since no boundary conditions can be prescribed at $r = 1$, contrary to the RTE and the $M_1^{1/2}$ solvers. M_1^+ is in its range of applicability and gives results almost as good as those of $M_1^{1/2}$. It might be presumed that the small difference between them comes from the difference of the boundary conditions. In all cases, P_1 gives mostly wrong results (for $\sigma = 0.01$, the anisotropic factor $f = \mathbf{F}_R/E_R$ found by P_1 at $r = 1.0$ is around 5, meaning a speed of light five times overestimated). Surprisingly, M_1 gives a good computation of the flux for all cases, but a rough computation of the energy.

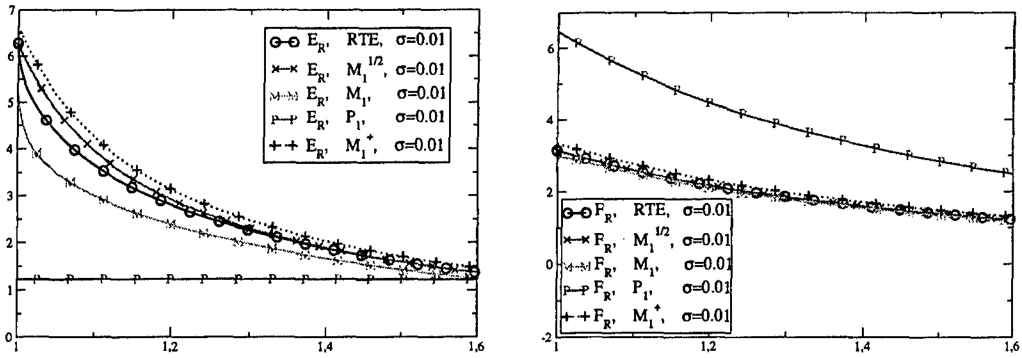

FIGURE 3. Comparison between 5 models for $\sigma = 0.01$. Left: Radiative Energy. Right: Radiative Flux.

These results valid the models and let presume that they will provide an accurate description of radiative fields of the solar atmosphere, which presents the same main characteristics of this simple geometry.

4.2. *Computation of the radiation field of the solar atmosphere*

The temperature and opacity profiles of the sun are taken from Stix (1989). These values are obtained from model C of Vernazza *et al.* (1981) and are representative of a quiet sun. The temperature and opacity profiles are truncated before the transition layer to the corona, which is here considered as a zero density zone. Confronted with the difficulty of determining mean absorption coefficients for the solar atmosphere, which are needed to model the opacity, we obtained the opacity from the optical depth τ at $500\,nm$ ($\sigma = \partial\tau_{500}/\partial r$).

The interior of the sun is assumed to be opaque enough to be represented by a black body emitting at $6910K$, which is the temperature at $25\,km$ below the surface. Outgoing boundary conditions are identical to the previous case. In Fig 4, the M_1^+ in spherical coordinates with spherical symmetry is solved and compared to the RTE ray tracing solution for the solar atmosphere.

The radiative temperature computed by M_1^+ is coincident with the one computed by the RTE solver close to the sun. Far away, the M_1^+ model finds a different plateau value (between $4960K$ and $4938K$) from the RTE solver (between $4700K$ and $4650K$). At $r = 6.97 \times 10^8\,m$, the error in the radiative temperature from the M_1^+ model, which finds $4950K$ instead of $4682K$, is 5.72%. At $r = 6.98 \times 10^8\,m$, the error is 6%. The radiative flux computed by both models presents, close to the sun, a small spike followed by a smooth bump close to the surface which is due to the opacity profile. The error of the flux computed by $M1^+$ is maximal at the top of the spike where it reaches 17%, while at the top of the bump the error is around 10%. Outside this highly variable zone, the profile of the radiative flux is computed with a precision of 5% by the M_1^+ model. Moreover, it should be noticed that, in this particular example, radiation is near isotropic ($f \simeq 0.5$), a condition in which the error introduced by M_1^+ is maximal, but it remains as we can see very admissible.

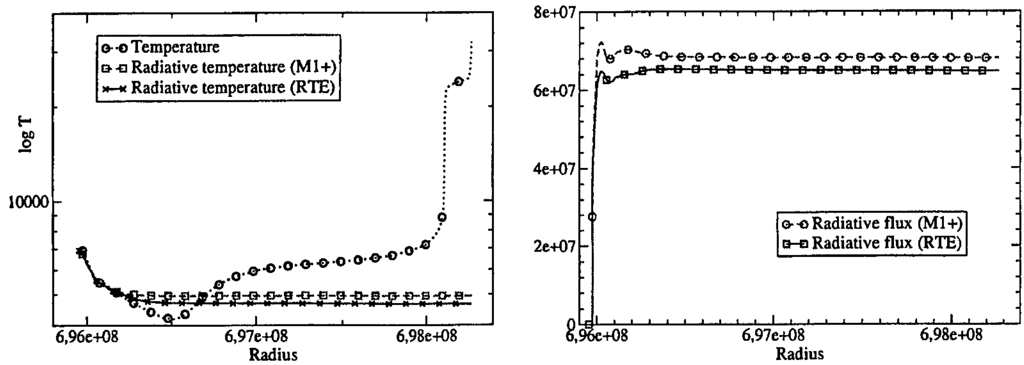

FIGURE 4. Radiation field of the sun atmosphere. Left: Radiative and matter Temperatures. Right: Radiative Flux.

5. Conclusion

A new three dimensional half moment model, called $M_1^{1/2}$, has been derived using a dynamic definition of the half quantities and the maximum entropy closure. The direction space is cut into two pieces according to the direction of propagation of the total flux, at each time and position. This splitting allows the radiative pressure to be closed, using the standard maximum entropy closure, with approximate constraints, but introduces non-trivial border terms.

By assuming that radiation is isotropic in the plane perpendicular to the flux, $P_\perp$, which seems admissible for a two-flux model, the radiative transfer equation has been integrated on these moving half-directional spaces. The integration is difficult but leads to small and simple expressions; the border terms account for the rotation needed to set the directional space in the direction of the flux. The value of the intensity in $P_\perp$ must be known, and a model for this quantity has been proposed.

An immediate application of this work has been presented: for the particular case of a three dimensional hot, opaque source emitting into a non re-emitting medium, the half moment model has been reduced to a new full moment model, called here M_1^+.

The model is here tested for the case of a hot one-dimensional Gaussian source radiating into media of various opacities. Very good agreement has been shown between the RTE and the $M_1^{1/2}$ model independently of the opacity. The relevance and the correctness of the closure has been shown, in one dimension, for a case theoretically similar to the solar atmosphere.

The radiation field of a solar atmosphere with an approximate opacity has been computed with the M_1^+ model. The results of M_1^+ are not as good as those presented in the previous theoretical problem: the error is always around 5% for all unknowns, except in a small transitional zone where the flux error reaches its maximal value of 17%. The solar problem close to the surface, being near isotropic, is a case for which the accuracy of M_1^+ should be lowest, but is nevertheless found to be admissible.

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